The spectral action for sub-Dirac operators

Yong Wang

Abstract

In this paper, for foliations with spin leaves, we compute the spectral action for sub-Dirac operators.

Keywords: sub-Dirac operators; spectral action ; Seely-dewitt coefficients

1 Introduction

Connes'spectral action principle ([Co]) in noncommutative geometry states that the physical action depends only on the spectrum. We assume that space-time is a product of a continuous manifold and a finite space. The spectral action is defined as the trace of an arbitrary function of the Dirac operator for the bosonic part and a Dirac type action of the fermionic part including all their interactions. In [CC1], Chamseddine and Connes computed the Spectral action for Dirac operators on spin manifolds and the Chamseddine-Connes spectral action comprises the Einsiein-Hilbert action of general relativity and the bosonic part of the action of the standard model of particle physics. In [HPS], Hanisch, Pfäffle and Stephan derived a formula for the gravitional part of the spectral action for Dirac operators on 4-dimensional spin manifolds with totally anti-symmetric torsion. They also deduced the Lagrangian for the Standard Model of particle Physics in the presence of torsion from the Chamseddine-Connes spectral action. In [CC2], Chamseddine and Connes studied the spectral action for spin manifolds with boundary and generalized this action to noncommutative spaces which are products of a spin manifold and a finite space. In [EILS],[ILS], the spectral actions for the noncommutative torus and $SU_q(2)$ are computed explicitly.

In this paper, we consider a compact foliation M with spin leaves. We don't assume that M is spin, so we have no Dirac operators on M, then we can not derive the physical action from the Chamseddine-Connes spectral action for Dirac operators. In [LZ], in order to prove the Connes' vanishing theorem for foliations with spin leaves, Liu and Zhang introduced sub-Dirac operators instead of Dirac operators. The sub-Dirac operator is a first order formally self adjoint elliptic differential operator. So we have a commutative spectral triple and we compute the spectral action for sub-Dirac operators.

This paper is organized as follows: In Section 2, we review the sub-Dirac operator and compute the spectral action for sub-Dirac operators. In Section 3, we compute the spectral action for sub-Dirac operators for the Standard Model.In Section 4, we

compute the spectral action for sub-Dirac operators for foliations with boundary.

2 The spectral action for sub-Dirac operators

Let (M,F) be a closed foliation and g^F be a metric on F. Let g^{TM} be a metric on TM which restricted to g^F on F. Let $F^\perp$ be the orthogonal complement of F in TM with respect to g^{TM}. Then we have the following orthogonal splitting,

$$TM = F \oplus F^\perp; \quad g^{TM} = g^F \oplus g^{F^\perp}, \tag{2.1}$$

where $g^{F^\perp}$ is the restriction of g^{TM} to $F^\perp$. Let $P, P^\perp$ be the orthogonal projection from TM to F,$F^\perp$ respectively. Let ∇^{TM} be the Levi-Civita connection of g^{TM} and ∇^F (resp. $\nabla^{F^\perp}$) be the restriction of ∇^{TM} to F (resp. $F^\perp$). That is,

$$\nabla^F = P\nabla^{TM}P, \quad \nabla^{F^\perp} = P^\perp\nabla^{TM}P^\perp. \tag{2.2}$$

We assume that F is oriented, spin and carries a fixed spin structure. We also assume that $F^\perp$ is oriented and that both $2p = \dim F$ and $q = \dim F^\perp$ are even.

Let $S(F)$ be the bundle of spinors associated to (F, g^F). For any $X \in \Gamma(F)$, denote by $c(X)$ the Clifford action of X on $S(F)$. Since $\dim F$ is even, we have the splitting $S(F) = S_+(F) \oplus S_-(F)$ and $c(X)$ exchanges $S_+(F)$ and $S_-(F)$.

Let $\wedge(F^{\perp,\star})$ be the exterior algebra bundle of $F^\perp$. Then $\wedge(F^{\perp,\star})$ carries a canonically induced metric $g^{\wedge(F^{\perp,\star})}$ from $g^{F^\perp}$. For any $U \in \Gamma(F^\perp)$, let $U^* \in \Gamma(F^{\perp,*})$ be the corresponding dual of U with respect to $g^{F^\perp}$. Now for $U \in \Gamma(F^\perp)$, set

$$c(U) = U^* \wedge -i_U, \quad \widehat{c}(U) = U^* \wedge +i_U, \tag{2.3}$$

where $U^*\wedge$ and i_U are the exterior and inner multiplication. Let $h_1.\cdots,h_q$ be an oriented local orthonormal basis of $F^\perp$. Then $\tau = (-\sqrt{-1})^{\frac{q(q+1)}{2}}c(h_1)\cdots c(h_q)$ and $\tau^2 = 1$. Now the $+1$ and -1 eigenspaces of τ give a splitting $\wedge(F^{\perp,\star}) = \wedge_+(F^{\perp,\star}) \oplus \wedge_-(F^{\perp,\star})$. Let $S(F)\widehat{\otimes}\wedge(F^{\perp,\star})$ be the $\mathbf{Z}_2$ graded tensor product of $S(F)$ and $\wedge(F^{\perp,\star})$. For $X \in \Gamma(F)$, $U \in \Gamma(F^\perp)$, the operators $c(X)$, $c(U)$, $\widehat{c}(U)$ extend naturally to $S(F)\widehat{\otimes}\wedge(F^{\perp,\star})$ and they are anticommute. The connections ∇^F, $\nabla^{F^\perp}$ lift to $S(F)$ and $\wedge(F^{\perp,\star})$ naturally. We write them $\nabla^{S(F)}$ and $\nabla^{\wedge(F^{\perp,\star})}$. Then $S(F)\widehat{\otimes}\wedge(F^{\perp,\star})$ carries the induced tensor product connection $\nabla^{S(F)\widehat{\otimes}\wedge(F^{\perp,\star})}$.

Let $S \in \Omega(T^*M) \otimes \Gamma(\mathrm{End}(TM))$ be defined by

$$\nabla^{TM} = \nabla^F + \nabla^{F^\perp} + S. \tag{2.4}$$

Then for any $X \in \Gamma(TM)$, $S(X)$ exchanges $\Gamma(F)$ and $\Gamma(F^\perp)$ and is skew-adjoint with respect to g^{TM}. Let V be a complex vector bundle with the metric connection ∇^V. Then $S(F)\widehat{\otimes}\wedge(F^{\perp,\star})\otimes V$ carries the induced tensor product connection $\nabla^{S(F)\widehat{\otimes}\wedge(F^{\perp,\star})\otimes V}$. Let $\{f_i\}_{i=1}^{2p}$ be an oriented orthonormal basis of F. Let

$$\widetilde{\nabla} = \nabla^{S(F)\widehat{\otimes}\wedge(F^{\perp,\star})} + \frac{1}{2}\sum_{j=1}^{2p}\sum_{s=1}^{q} < S(.)f_j, h_s > c(f_j)c(h_s)$$

$$\widetilde{\nabla}^{F,V} = \widetilde{\nabla} \otimes \mathrm{Id}_V + \mathrm{Id}_{S(F)\widehat{\otimes}\wedge(F^{\perp,\star})} \otimes \nabla^V. \tag{2.5}$$

Since the vector bundle $F^{\perp}$ might well be non-spin, Liu and Zhang [LZ] introduced the following sub-Dirac operator:

Definition 2.1 Let $D_{F,V}$ be the operator mapping from $\Gamma(S(F)\widehat{\otimes}\wedge(F^{\perp,\star})\otimes V)$ to itself defined by

$$D_{F,V} = \sum_{i=1}^{2p} c(f_i)\widetilde{\nabla}^{F,V}_{f_i} + \sum_{s=1}^{q} c(h_s)\widetilde{\nabla}^{F,V}_{h_s}. \tag{2.6}$$

Let $\triangle^{F,V}$ be the Bochner Laplacian defined by

$$\triangle^{F,V} := -\sum_{i=1}^{2p}(\widetilde{\nabla}^{F,V}_{f_i})^2 - \sum_{s=1}^{q}(\widetilde{\nabla}^{F,V}_{h_s})^2 + \widetilde{\nabla}^{F,V}_{\sum_{i=1}^{2p}\nabla^{TM}_{f_i}f_i} + \widetilde{\nabla}^{F,V}_{\sum_{s=1}^{q}\nabla^{TM}_{h_s}h_s}. \tag{2.7}$$

Let r_M be the scalar curvature of the metric g^{TM}. Let $R^{F^{\perp}}$ and R^V be curvature of $F^{\perp}$ and V. Then we have the following Lichnerowicz formula for $D_{F,V}$.

Theorem 2.2([LZ]) *The following identity holds*

$$\begin{aligned}
D_{F,V}^2 &= \triangle^{F,V} + \frac{1}{2}\sum_{i,j=1}^{2p} c(f_i)c(f_j)R^V(f_i,f_j)\\
&+\sum_{i=1}^{2p}\sum_{s=1}^{q} c(f_i)c(h_s)R^V(f_i,h_s) + \frac{1}{2}\sum_{s,t=1}^{q} c(h_s)c(h_t)R^V(h_s,h_t)\\
&+\frac{r_M}{4} + \frac{1}{4}\sum_{i=1}^{2p}\sum_{r,s,t=1}^{q}\left\langle R^{F^{\perp}}(f_i,h_r)h_t,h_s\right\rangle c(f_i)c(h_r)\widehat{c}(h_s)\widehat{c}(h_t)\\
&+\frac{1}{8}\sum_{i,j=1}^{2p}\sum_{s,t=1}^{q}\left\langle R^{F^{\perp}}(f_i,f_j)h_t,h_s\right\rangle c(f_i)c(f_j)\widehat{c}(h_s)\widehat{c}(h_t)\\
&+\frac{1}{8}\sum_{s,t,r,l=1}^{q}\left\langle R^{F^{\perp}}(h_r,h_l)h_t,h_s\right\rangle c(h_r)c(h_l)\widehat{c}(h_s)\widehat{c}(h_t).
\end{aligned} \tag{2.8}$$

When V is a complex line bundle, we write D_F instead of $D_{F,E}$. For the sub-Dirac operator D_F we will calculate the bosonic part of the spectral action. It is defined to be the number of eigenvalues of D_F in the interval $[-\wedge,\wedge]$ with $\wedge\in\mathbf{R}^+$. As in [CC1], it is expressed as

$$I = \mathrm{tr}\widehat{F}\left(\frac{D_F^2}{\wedge^2}\right).$$

Here tr denotes the operator trace in the L^2 completion of $\Gamma(S(F)\widehat{\otimes}\wedge(F^{\perp,\star}))$, and $\widehat{F}:\mathbf{R}^+\to\mathbf{R}^+$ is a cut-off function with support in the interval $[0,1]$ which is constant

near the origin. Let $\dim M = m$. By Theorem 2.2, we have the heat trace asymptotics for $t \to 0$,

$$\mathrm{tr}(e^{-tD_F^2}) \sim \sum_{n\geq 0} t^{n-\frac{m}{2}} a_{2n}(D_F^2).$$

One uses the Seely-deWitt coefficients $a_{2n}(D_F^2)$ and $t = \wedge^{-2}$ to obtain an asymptotics for the spectral action when $\dim M = 4$ [CC1]

$$I = \mathrm{tr}\widehat{F}\left(\frac{D_F^2}{\wedge^2}\right) \sim \wedge^4 F_4 a_0(D_F^2) + \wedge^2 F_2 a_2(D_F^2) + \wedge^0 F_0 a_4(D_F^2) \quad \text{as} \quad \wedge \to \infty \tag{2.9}$$

with the first three moments of the cut-off function which are given by $F_4 = \int_0^\infty s\widehat{F}(s)ds$, $F_2 = \int_0^\infty \widehat{F}(s)ds$ and $F_0 = \widehat{F}(0)$. Let

$$-E = \frac{r_M}{4} + W = \frac{r_M}{4} + \frac{1}{4}\sum_{i=1}^{2p}\sum_{r,s,t=1}^{q}\left\langle R^{F^\perp}(f_i, h_r)h_t, h_s\right\rangle c(f_i)c(h_r)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{1}{8}\sum_{i,j=1}^{2p}\sum_{s,t=1}^{q}\left\langle R^{F^\perp}(f_i, f_j)h_t, h_s\right\rangle c(f_i)c(f_j)\widehat{c}(h_s)\widehat{c}(h_t)$$

$$+\frac{1}{8}\sum_{s,t,r,l=1}^{q}\left\langle R^{F^\perp}(h_r, h_l)h_t, h_s\right\rangle c(h_r)c(h_l)\widehat{c}(h_s)\widehat{c}(h_t), \tag{2.10}$$

and

$$\Omega_{ij} = \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j} - \widetilde{\nabla}_{e_j}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{[e_i,e_j]}, \tag{2.11}$$

where e_i is f_i or h_s. We use [G, Thm 4.1.6] to obtain the first three coefficients of the heat trace asymptotics:

$$a_0(D_F) = (4\pi)^{-\frac{m}{2}}\int_M \mathrm{tr}(\mathrm{Id})\mathrm{dvol}, \tag{2.12}$$

$$a_2(D_F) = (4\pi)^{-\frac{m}{2}}\int_M \mathrm{tr}[(r_M + 6E)/6]dvol, \tag{2.13}$$

$$a_4(D_F) = \frac{(4\pi)^{-\frac{m}{2}}}{360}\int_M \mathrm{tr}[-12R_{ijij,kk} + 5R_{ijij}R_{klkl}$$

$$-2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{,kk} + 30\Omega_{ij}\Omega_{ij}]dvol. \tag{2.14}$$

Since $\dim[S(F)\widehat{\otimes}\wedge(F^{\perp,\star})] = 2^{p+q}$ and $m = 2p + q$, then we have $a_0(D_F) = \frac{1}{2^p\pi^{p+\frac{q}{2}}}\int_M dvol$. By Clifford relations and cyclicity of the trace and the trace of the odd degree operator being zero, we have

$$\mathrm{tr}(c(f_i)) = 0;\ \mathrm{tr}(c(f_i)c(f_j)) = 0 \text{ for } i \neq j;$$

$$\mathrm{tr}(c(h_r)c(h_l)\widehat{c}(h_s)\widehat{c}(h_t)) = 0,\ \text{for } r \neq l. \tag{2.15}$$

and

$$\mathrm{tr}E = -2^{p+q} \cdot \frac{r_M}{4}, \quad a_2(D_F) = -\frac{1}{12 \cdot 2^p \pi^{p+\frac{q}{2}}} \int_M r_M dvol. \tag{2.16}$$

Let I_1, I_2, I_3 denote respectively the last three terms in (2.10). By (2.15), we have

$$\mathrm{tr}(E^2) = \mathrm{tr}(\frac{r_M^2}{16} + W^2) = \mathrm{tr}(\frac{r_M^2}{16} + I_1^2 + I_2^2 + I_3^2). \tag{2.17}$$

$$\begin{aligned}\mathrm{tr}(I_1^2) = \frac{1}{16} \sum_{i,i'=1}^{2p} \sum_{r,r',s,s',t,t'=1}^{q} &\left\langle R^{F^\perp}(f_i, h_r)h_t, h_s \right\rangle \left\langle R^{F^\perp}(f_{i'}, h_{r'})h_{t'}, h_{s'} \right\rangle \\ &\cdot \mathrm{tr}[c(f_i)c(h_r)\widehat{c}(h_s)\widehat{c}(h_t)c(f_{i'})c(h_{r'})\widehat{c}(h_{s'})\widehat{c}(h_{t'})]\end{aligned} \tag{2.18}$$

Similar to (2.15), we have

$$\begin{aligned}&\mathrm{tr}[c(f_i)c(h_r)\widehat{c}(h_s)\widehat{c}(h_t)c(f_{i'})c(h_{r'})\widehat{c}(h_{s'})\widehat{c}(h_{t'})] \\ &= -\delta_i^{i'} \delta_r^{r'} 2^p \mathrm{tr}_{\wedge(F^\perp,\star)}[\widehat{c}(h_s)\widehat{c}(h_t)\widehat{c}(h_{s'})\widehat{c}(h_{t'})]\end{aligned} \tag{2.19}$$

Since $t \neq s, \ t' \neq s'$. we get

$$\mathrm{tr}_{\wedge(F^\perp,\star)}[\widehat{c}(h_s)\widehat{c}(h_t)\widehat{c}(h_{s'})\widehat{c}(h_{t'})] = (\delta_t^{s'} \delta_s^{t'} - \delta_t^{t'} \delta_s^{s'})2^q \tag{2.20}$$

By (2.19) and (2.20), we have

$$\mathrm{tr}(I_1^2) = \frac{2^{p+q}}{8} \sum_{i=1}^{2p} \sum_{r,s,t=1}^{q} \left\langle R^{F^\perp}(f_i, h_r)h_t, h_s \right\rangle^2. \tag{2.21}$$

Similarly we have

$$\mathrm{tr}(I_2^2) = \frac{2^{p+q}}{16} \sum_{i,j=1}^{2p} \sum_{s,t=1}^{q} \left\langle R^{F^\perp}(f_i, f_j)h_t, h_s \right\rangle^2; \tag{2.22}$$

$$\mathrm{tr}(I_3^2) = \frac{2^{p+q}}{16} \sum_{s,t,r,l=1}^{q} \left\langle R^{F^\perp}(h_r, h_l)h_t, h_s \right\rangle^2. \tag{2.23}$$

So we get

$$\mathrm{trE}^2 = \frac{2^{p+q}}{16} r_M^2 + \frac{2^{p+q}}{16} ||R^{F^\perp}||^2, \tag{2.24}$$

where

$$\begin{aligned}||R^{F^\perp}||^2 = 2 \sum_{i=1}^{2p} \sum_{r,s,t=1}^{q} \left\langle R^{F^\perp}(f_i, h_r)h_t, h_s \right\rangle^2 \\ + \sum_{i,j=1}^{2p} \sum_{s,t=1}^{q} \left\langle R^{F^\perp}(f_i, f_j)h_t, h_s \right\rangle^2 + \sum_{s,t,r,l=1}^{q} \left\langle R^{F^\perp}(h_r, h_l)h_t, h_s \right\rangle^2.\end{aligned} \tag{2.25}$$

Nextly we compute $\mathrm{tr}[\Omega_{ij}\Omega_{ij}]$ in a local coordinate, so we can assume that M is spin and $\widetilde{\widetilde{\nabla}}$ is the standard twisted connection on the twisted spinors bundle $S(TM)\otimes S(F^{\perp})$. Then

$$\Omega_{ij}=R^{S(TM)}(e_i,e_j)\otimes \mathrm{Id}_{S(F^{\perp})}+\mathrm{Id}_{S(TM)}\otimes R^{S(F^{\perp})}(e_i,e_j)$$

$$=-\frac{1}{4}R^M_{ijkl}c(e_k)c(e_l)\otimes \mathrm{Id}_{S(F^{\perp})}-\frac{1}{4}\mathrm{Id}_{S(TM)}\otimes \left\langle R^{F^{\perp}}(e_i,e_j)h_s,h_t\right\rangle c(h_s)c(h_t). \quad (2.26)$$

Similar to the computations of trE^2, we get

$$\mathrm{tr}[\Omega_{ij}\Omega_{ij}]=-\frac{2^{p+q}}{8}(R^2_{ijkl}+||R^{F^{\perp}}||^2) \quad (2.27)$$

By the divergence theorem and (2.24) and (2.27), we have

$$a_4(D_F^2)=\frac{1}{360\cdot 2^p\pi^{p+\frac{q}{2}}}\int_M\left(\frac{5}{4}r_M^2-2R_{ijik}R_{ljlk}-\frac{7}{4}R^2_{ijkl}+\frac{15}{2}||R^{F^{\perp}}||^2\right)dvol. \quad (2.28)$$

3 The spectral action for the Standard Model associated to sub-Dirac operators

In this section, we let $m=4$. We consider the product space $\mathcal{H}$ of the L^2 completion of $\Gamma(S(F)\widehat{\otimes}\wedge(F^{\perp,\star}))$ and a finite dimensional Hilbert space $\mathcal{H}_f$ (called internal Hilbert space). The specific particle model is encoded in $\mathcal{H}_f$. On the bundle $S(F)\widehat{\otimes}\wedge(F^{\perp,\star})\otimes\mathcal{H}_f$ one considers a connection $\widetilde{\nabla}^{F,\mathcal{H}_f}$ in (2.5) and $\nabla^{\mathcal{H}_f}$ is a covariant derivative in the trivial bundle $\mathcal{H}_f$ induced gauge fields. The associated Dirac operator to $\widetilde{\nabla}^{F,\mathcal{H}_f}$ is called D_F^f. The generalized Dirac operator of the Standard Model $D_{F,\Phi}$ contains the Higgs boson, Yukawa couplings, neutrino masses and the CKM-matrix encoded in a field Φ of endomorphisms of $\mathcal{H}_f$. We define $D_{F,\Phi}$ for sections $\psi\otimes\chi\in\mathcal{H}$ as

$$D_{F,\Phi}(\psi\otimes\chi)=D_F^f(\psi\otimes\chi)+\gamma_5\psi\otimes\Phi\chi, \quad (3.1)$$

where $\gamma_5=e_0e_1e_2e_3$ is the volume element. We choose the same Φ as Φ in [CC1]. The bosonic part of the Lagrangian of the Standard Model is obtained by replacing D_F by $D_{F,\Phi}$ in (2.9). In (2.8), we write $D^2_{F,\mathcal{H}_f}=\triangle^{F,\mathcal{H}_f}+W_1$. Then direct computations show

$$D^2_{F,\Phi}=\triangle^{F,\mathcal{H}_f}-E_\Phi, \quad (3.2)$$

where the potential is defined as

$$E_\Phi(\psi\otimes\chi)=-W_1(\psi\otimes\chi)+\sum_{i=1}^4\gamma_5c(e_i)\cdot\psi\otimes[\nabla^{H_f}_{e_i},\Phi]\chi-\psi\otimes\Phi^2\chi. \quad (3.3)$$

We denote the trace on $\mathcal{H}$ and on $\mathcal{H}_f$ as Tr and tr_f. From (3.3), we have

$$\mathrm{Tr}(E_\Phi)=\mathrm{dim}\mathcal{H}_f\cdot 2^{p+q-2}r_M-2^{p+q}\mathrm{tr}_f(\Phi^2). \quad (3.4)$$

For Seely-deWitt coefficient $a_4(D^2_{F,\Phi})$ we also need to calculate

$$(E_\Phi)^2(\psi\otimes\chi)=W_1^2(\psi\otimes\chi)+\sum_{i,j=1}^{4}\gamma_5c(e_i)\gamma_5c(e_j)\cdot\psi\otimes[\nabla^{H_f}_{e_i},\Phi][\nabla^{H_f}_{e_j},\Phi]\chi$$

$$+\psi\otimes\Phi^4\chi-2E\psi\otimes\Phi^2\chi+\frac{1}{2}\sum_{i,j=1}^{2p}c(f_i)c(f_j)\psi\otimes[\Phi^2R^{\mathcal{H}_f}(f_i,f_j)+R^{\mathcal{H}_f}(f_i,f_j)\Phi^2]\chi$$

$$+\sum_{i=1}^{2p}\sum_{s=1}^{q}c(f_i)c(h_s)\psi\otimes[\Phi^2R^{\mathcal{H}_f}(f_i,h_s)+R^{\mathcal{H}_f}(f_i,h_s)\Phi^2]\chi$$

$$+\frac{1}{2}\sum_{s,t=1}^{q}c(h_s)c(h_t)\psi\otimes[\Phi^2R^{\mathcal{H}_f}(h_s,h_t)+R^{\mathcal{H}_f}(h_s,h_t)\Phi^2]\chi$$

$$-\sum_{i=1}^{4}\gamma_5c(e_i)\psi\otimes(\Phi^2[\nabla^{H_f}_{e_i},\Phi]+[\nabla^{H_f}_{e_i},\Phi]\Phi^2)\chi$$

$$+\sum_{i=1}^{4}(E\gamma_5c(e_i)\psi+\gamma_5c(e_i)E\psi)\otimes[\nabla^{H_f}_{e_i},\Phi]\chi$$

$$-\frac{1}{2}\sum_{i,j,k=1}^{4}\gamma_5c(e_i)c(e_j)c(e_k)\psi\otimes[\nabla^{H_f}_{e_i},\Phi]R^{\mathcal{H}_f}(e_j,e_k)\chi$$

$$-\frac{1}{2}\sum_{i,j,k=1}^{4}c(e_j)c(e_k)\gamma_5c(e_i)\psi\otimes R^{\mathcal{H}_f}(e_j,e_k)[\nabla^{H_f}_{e_i},\Phi]\chi.\qquad(3.5)$$

By Clifford relations and cyclicity of the trace and the trace of the odd degree operator being zero, only the first four summands on the right-hand side contribute to the trace of $(E_\Phi)^2$. By direct computations, we get

$$\mathrm{Tr}(E_\Phi^2)=\dim\mathcal{H}_f\frac{2^{p+q}}{16}(r_M^2+||R^{F^\perp}||^2)-2^{p+q-1}\sum_{i,j=1}^{4}\mathrm{tr}_f(\Omega^f_{ij}\Omega^f_{ij})$$

$$+2^{p+q-1}r_M\mathrm{tr}_f(\Phi^2)+2^{p+q}\mathrm{tr}_f(\Phi^4)+2^{p+q}\sum_{i=1}^{4}\mathrm{tr}_f([\nabla^{H_f}_{e_i},\Phi]^2).\qquad(3.6)$$

By (2.27), we have

$$\mathrm{Tr}(\widetilde{\Omega}^f_{ij}\widetilde{\Omega}^f_{ij})=-\dim\mathcal{H}_f\cdot\frac{2^{p+q}}{8}(R^2_{ijkl}+||R^{F^\perp}||^2)+2^{p+q}\mathrm{tr}_f(\Omega^f_{ij}\Omega^f_{ij}).\qquad(3.7)$$

We choose the finite space $\mathcal{H}_f$ according to the construction of the noncommutative Standard Model [CC1], $\dim\mathcal{H}_f=96$ and $\nabla^{\mathcal{H}_f}$ is the appropriate covariant derivative

associated to the Standard Model gauge group $U(1)_Y\times SU(2)_\omega\times SU(3)_c$. We know that (for related notations see [HPS], [IKS]),

$$\mathrm{tr}_f(\Omega^f_{ij}\Omega^f_{ij})=\frac{48}{5}g_3^2||G||^2+\frac{48}{5}g_2^2||F_1||^2+16g_1^2||B||^2, \tag{3.8}$$

$$\mathrm{tr}_f([\nabla^{H_f}_{e_i},\Phi]^2)=4a|D_\nu\varphi|^2,\quad \mathrm{tr}_f(\Phi^2)=4a|\phi|^2+2c,\quad \mathrm{tr}_f(\Phi^4)=4b|\phi|^4+8e|\phi|^2+2d. \tag{3.9}$$

Then we get

$$a_0(D_{F,\Phi})=\frac{96}{2^p\pi^{p+\frac{q}{2}}}\int_M dvol, \tag{3.10}$$

$$a_2(D_{F,\Phi})=\frac{1}{2^p\pi^{p+\frac{q}{2}}}\int_M(40r_M-4a|\phi|^2-2c)dvol, \tag{3.11}$$

$$a_4(D_{F,\Phi})=\frac{1}{360\cdot 2^p\pi^{p+\frac{q}{2}}}\int_M\big\{4000r_M^2-192R_{ijik}R_{ljlk}-168R^2_{ijkl}+120ar_M|\varphi|^2$$
$$+60cr_M+720||R^{F^\perp}||^2-576g_3^2||G||^2-576g_2^2||F_1||^2-960g_1^2||B||^2$$
$$+720b|\varphi|^4+1440e|\varphi|^2+360d+720|D_\nu\varphi|^2\big\}\,dvol. \qquad (3.12)dvol$$

In presence of the Standard Model fields we obtain essentially one new term (apart from the usual suspects)

$$I_{\mathrm{new}}=\frac{2}{2^p\pi^{p+\frac{q}{2}}}\int_M||R^{F^\perp}||^2dvol. \tag{3.13}$$

4 The spectral action for foliations with boundary

In this section, we let M be a foliation with boundary ∂M. Let $\psi\in\Gamma(S(F)\widehat{\otimes}\wedge(F^{\perp,\star})$, We impose the Dirichlet boundary conditions $\psi|_{\partial M}=0$. With the Dirichlet boundary conditions, we have the heat trace asymptotics for $t\to 0$ [BG],

$$\mathrm{tr}(e^{-tD_F^2})\sim\sum_{n\geq 0}t^{\frac{n-m}{2}}a_n(D_F^2).$$

One uses the Seely-deWitt coefficients $a_n(D_F^2)$ and $t=\wedge^{-2}$ to obtain an asymptotics for the spectral action when $\mathrm{dim}M=4$ [ILV (18)]

$$I=\mathrm{tr}\widehat{F}\left(\frac{D_F^2}{\wedge^2}\right)\sim\wedge^4F_4a_0(D_F^2)+\wedge^3F_3a_1(D_F^2)$$
$$+\wedge^2F_2a_2(D_F^2)+\wedge F_1a_3(D_F^2)+\wedge^0F_0a_4(D_F^2)\ \text{ as }\ \wedge\to\infty \tag{4.1}$$

where $F_k:=\frac{1}{\Gamma(\frac{k}{2})}\int_0^\infty\widehat{F}(s)s^{\frac{k}{2}-1}ds$. Let $N=e_m$ be the inward pointing unit normal vector on ∂M and $e_i, 1\leq i\leq m-1$ be the orthonormal frame on $T(\partial M)$. Let $L_{ab}=(\nabla_{e_a}e_b,N)$ be the second fundamental form and indices $\{a,b,\cdots\}$ range from 1

through $m-1$. We use [BG, Thm 1.1] to obtain the first five coefficients of the heat trace asymptotics:

$$a_0(D_F) = (4\pi)^{-\frac{m}{2}} \int_M \mathrm{tr}(\mathrm{Id}) \mathrm{dvol_M}, \tag{4.2}$$

$$a_1(D_F) = -4^{-1}(4\pi)^{-\frac{(m-1)}{2}} \int_{\partial M} \mathrm{tr}(\mathrm{Id}) \mathrm{dvol_{\partial M}}, \tag{4.3}$$

$$a_2(D_F) = (4\pi)^{-\frac{m}{2}} 6^{-1} \{ \int_M \mathrm{tr}(r_M + 6E) dvol_M + 2 \int_{\partial M} \mathrm{tr}(L_{aa}) dvol_{\partial M} \}, \tag{4.4}$$

$$\begin{aligned} a_3(D_F) = -4^{-1}(4\pi)^{-\frac{(m-1)}{2}} 96^{-1} \{ \int_{\partial M} \mathrm{tr}(96E + 16r_M \\ +8R_{aNaN} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab}) dvol_{\partial M}) \}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} a_4(D_F) = \frac{(4\pi)^{-\frac{m}{2}}}{360} \{ \int_M \mathrm{tr}[-12R_{ijij,kk} + 5R_{ijij}R_{klkl} \\ -2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{,kk} + 30\Omega_{ij}\Omega_{ij}] dvol_M \\ + \int_{\partial M} \mathrm{tr}(-120E;_N - 18r_M;_N + 120EL_{aa} + 20r_ML_{aa} + 4R_{aNaN}L_{bb} - 12R_{aNbN}L_{ab} \\ +4R_{abcd}L_{ac} + 24L_{aa;bb} + 40/21L_{aa}L_{bb}L_{cc} - 88/7L_{ab}L_{ab}L_{cc} + 320/21L_{ab}L_{bc}L_{ac}) dvol_{\partial M} \}. \end{aligned} \tag{4.6}$$

By (2.16) and (2.28) and the divergence theorem for manifolds with boundary, we get

$$a_0(D_F) = \frac{1}{2^p \pi^{p+\frac{q}{2}}} \int_M dvol_M, \tag{4.7}$$

$$a_1(D_F) = -4^{-1}(4\pi)^{-\frac{(m-1)}{2}} 2^{p+q} \int_{\partial M} dvol_{\partial M}, \tag{4.8}$$

$$a_2(D_F) = \frac{1}{12 \cdot 2^p \pi^{p+\frac{q}{2}}} (- \int_M r_M dvol_M + 4 \int_{\partial M} L_{aa} dvol_{\partial M}), \tag{4.9}$$

$$\begin{aligned} a_3(D_F) = -4^{-1}(4\pi)^{-\frac{(m-1)}{2}} 96^{-1} 2^{p+q} \{ \int_{\partial M} (-8r_M \\ +8R_{aNaN} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab}) dvol_{\partial M} \}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} a_4(D_F) = \frac{(4\pi)^{-\frac{m}{2}}}{360} 2^{p+q} \{ \int_M \left(\frac{5}{4} r_M^2 - 2R_{ijik}R_{ljlk} - \frac{7}{4} R_{ijkl}^2 + \frac{15}{2} ||R^{F^\perp}||^2 \right) dvol_M \\ + \int_{\partial M} \mathrm{tr}(-51r_{M;N} - 10r_ML_{aa} + 4R_{aNaN}L_{bb} - 12R_{aNbN}L_{ab} \\ +4R_{abcd}L_{ac} + 24L_{aa;bb} + 40/21L_{aa}L_{bb}L_{cc} - 88/7L_{ab}L_{ab}L_{cc} + 320/21L_{ab}L_{bc}L_{ac}) dvol_{\partial M} \}. \end{aligned} \tag{4.11}$$

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School of Mathematics and Statistics, Northeast Normal University, Changchun Jilin, 130024, China
E-mail: *wangy581@nenu.edu.cn*